MATH 830 FALL 2025: HOMEWORK 3

For this assignment you will work in pairs as on the previous assignment. These solutions should be typeset using LaTex. Each member of the team must contribute both to the solutions and the typesetting. As before, for this assignment, you may use your notes, the Daily Summary, and any daily homework you have done. You may not consult outside sources, including, any algebra textbook, the internet, graduate students not in this class, or any professor except your Math 830 instructor. You may not cite - without proof - any facts not covered in class or the homework. To receive full credit, all proofs must be complete and contain the appropriate amount of detail. Hard copies of each team's solutions are due in pdf format on Monday, November 17.

Throughout R will denote a commutative ring and M will denote an R-module.

1. Let $\phi : R^n \to R^m$ be an R-module homomorphism. First show that there is an $m \times n$ matrix A such that $\phi(v) = Av$, for all $v \in R^n$. Here were are writing the elements of R^n and R^m as column vectors. Then show that the induced map $\phi^* : \mathrm{Hom}_R(R^m, R) \to \mathrm{Hom}_R(R^n, R)$ is multiplication by A^t, the transpose of A. Use these facts to calculate $\mathrm{Ext}^n_R(R/I, R)$, for all $n \geq 0$, for $R := k[x, y, z]/(xy - z^2)$, and $I = (x, z)R$.

Solution. The first part is just like in linear algebra: If $e_1, \ldots, e_n \in R^n$ is the standard basis and $\phi(e_i) = C_i$, a column vector in R^m, then if we let A be the $m \times n$ matrix whose columns are $C_1, \ldots, C_n$, then $\phi(v) = Av$, for all column vectors $v \in R^n$. Let us write $A := (a_{ij})$.

For the second part, we let $e_1^*, \ldots, e_n^*$ be the dual basis for $\mathrm{Hom}_R(R^n, R)$ and $f_1^*, \ldots, f_m^*$ be the dual basis for $\mathrm{Hom}_R(R^m, R)$, where $f_1, \ldots, f_m \in R^m$ is the standard basis. Here the dual basis means the same as in linear algebra $e_i^*(e_j) = \delta_{ij}$, the Kronecker delta. Now consider $\phi^*(f_j^*)$. We need to write this in terms of $e_1^*, \ldots, e_n^*$. By definition, $\phi^*((f_j^*) = f_j^* \circ \phi$ as a map from R^n to R. Then

$$f_j^*(\phi(e_i)) = f_j^*(a_{1i}f_1 + \cdots + a_{mi}f_m) = a_{ji}.$$

It follows that $f_j^* \circ \phi = a_{j1}e_1^* + \cdots + a_{jn}e_n^*$, since these functionals agree on $e_1, \ldots, e_n$. In other words, $\phi(f_j^*) = a_{j1}e_1^* + \cdots + a_{jn}e_n^*$. Therefore the matrix of ϕ^* with respect to the bases $\{e_1^*, \ldots, e_n^*\}$ and $\{f_1^*, \ldots, f_m^*\}$ has the jth row of A as its jth column, which means this matrix is just A^t. When we identify $\mathrm{Hom}_R(R^n, R)$ with R^n, we are identifying each e_i^* with e_i, and similarly when we identify $\mathrm{Hom}_R(R^m, R)$ with R^m, so that with these identifications, ϕ^* is multiplication by A^t.

For the final statement, it turns out that once we know the relations on x, z as elements of R, all further relations in the resolution of R/I and the computation of cohomology in its dual are easily identified. Suppose $a + bz \equiv 0$ in R, then $ax + bz = c(xy - z^2)$ in $S := k[x, y, z]$. Thus, $(a - cy)x + (b + cz)z = 0$ in S. Therefore $\begin{pmatrix} a - cy \\ b + cz \end{pmatrix} - = d\begin{pmatrix} -z \\ x \end{pmatrix}$, so that $\begin{pmatrix} a \\ b \end{pmatrix} = d\begin{pmatrix} -z \\ x \end{pmatrix} + c\begin{pmatrix} y \\ -z \end{pmatrix}$. Thus the relations on x, z are contained in the submodule generated by the columns of $\begin{pmatrix} -z & y \\ x & -z \end{pmatrix}$. On the other hand, the columns of this matrix are clearly relations on x and z, the start of a free resolution of R/I is

$$R^2 \xrightarrow{\begin{pmatrix} -z & y \\ x & -z \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & z \end{pmatrix}} R \to R/I \to 0.$$

To calculate the kernel of multiplication by $A := \begin{pmatrix} -z & y \\ x & -z \end{pmatrix}$, we note that if $\begin{pmatrix} -z & y \\ x & -z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then $ax + (-b)z \equiv 0$ in R, so that $\begin{pmatrix} a \\ -b \end{pmatrix}$ is in the column space of $\begin{pmatrix} -z & y \\ x & -z \end{pmatrix}$. Thus, $\begin{pmatrix} a \\ b \end{pmatrix}$ is in the column space of $B := \begin{pmatrix} z & y \\ x & z \end{pmatrix}$. It is easy to check that $AB = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, thus, the free resolution of R/I continues as

$$\cdots \longrightarrow R^2 \xrightarrow{B} R^2 \xrightarrow{A} R^2 \xrightarrow{\begin{pmatrix} x & z \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

Continuing in this way we see that we obtain a periodic resolution of R/I

$$\cdots \longrightarrow R^2 \xrightarrow{B} R^2 \xrightarrow{A} R^2 \xrightarrow{B} R^2 \xrightarrow{A} R^2 \xrightarrow{\begin{pmatrix} x & z \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

Applying $\mathrm{Hom}_R(-, R)$ and dropping R/I, we seek the homology of the complex

$$0 \to R \xrightarrow{\begin{pmatrix} x \\ z \end{pmatrix}} R^2 \xrightarrow{A^t} R^2 \xrightarrow{B^t} R^2 \xrightarrow{A^t} R^2 \xrightarrow{B^t} \cdots$$

Since R is an integral domain, and the first map (on the left) in the complex above takes r to $\begin{pmatrix} rx \\ rz \end{pmatrix}$, the kernel of this map is zero, so $\mathrm{Ext}^0_R(R/I, R) = 0$. Calculating as above, it is easy to see that the kernel of multiplication by A^t is the image of multiplication by B^t and the kernel of multiplication by B^t is the image of multiplication by A^t. This shows $\mathrm{Ext}_R^n(R/I,R)=0$, for $n \geq 0$. This also gives that $\mathrm{Ext}_R^1(R/I,R)$ is the cyclic module $L := \langle \begin{pmatrix} z \\ y \end{pmatrix}, \begin{pmatrix} x \\ z \end{pmatrix}\rangle / \langle \begin{pmatrix} x \\ z \end{pmatrix}\rangle$. Now $x \cdot \begin{pmatrix} z \\ y \end{pmatrix} \equiv z \cdot \begin{pmatrix} x \\ z \end{pmatrix}$ and $z \cdot \begin{pmatrix} z \\ y \end{pmatrix} \equiv y \cdot \begin{pmatrix} x \\ z \end{pmatrix}$, which shows that I annihilates L. On the other hand, if $t \cdot L \equiv 0$, then $tz \equiv sx$, for some $s \in R$, and our relations on x, z show that $t \in I$. Thus, $\mathrm{ann}(L) = I$, giving $\mathrm{Exdt}_R^1(R/I,R) \cong R/I$.

2. Prove the injective version of the Ext Lemma. Namely: Given an exact sequence $0 \to N \xrightarrow{i} Q \xrightarrow{\pi} C \to 0$, with Q injective, then for any R-modules A, there exists an exact sequence

$$0 \to \mathrm{Hom}_R(A,N) \xrightarrow{\tilde{i}} \mathrm{Hom}_R(A,Q) \xrightarrow{\tilde{\pi}} \mathrm{Hom}_R(A,C) \to \mathrm{Ext}_R^1(A,N) \to 0.$$

Moreover, for all $n \geq 1$, $\mathrm{Ext}_R^n(A,C) \cong \mathrm{Ext}_R^{n+1}(A,N)$. You may assume the equivalence in Proposition-Definition from Wednesday, October 8 hold.

Solution. Since $0 \to \mathrm{Hom}_R(A,N) \xrightarrow{\tilde{i}} \mathrm{Hom}_R(A,Q) \xrightarrow{\tilde{\pi}} \mathrm{Hom}_R(A,C)$ is exact, we just need a surjective R-module homomorphism $g : \mathrm{Hom}_R(A,C) \to \mathrm{Ext}_R^1(A,N)$ whose kernel is the image of $\tilde{\pi}$. Towards this end, let $0 \to N \xrightarrow{i} Q \xrightarrow{\psi_1} Q_2 \xrightarrow{\psi_2} Q_3$ be the start of an injective resolution of N, so that the image of ψ_1 (and hence the kernel of ψ_2) is C. Write $j : C \to Q_1$ for the natural inclusion. Now suppose $h \in \mathrm{Hom}_R(A,C)$. Then, $jh \in \mathrm{Hom}_R(A,Q_1)$. If we show $\psi_2 jh = 0$, then $jh \in \ker(\tilde{\psi_2})$, and we can define $g(h) := [jh]$, the class of jh in $\ker(\psi_2)/\mathrm{im}(\psi_1) = \mathrm{Ext}_R^1(Q,N)$. Suppose $a \in A$. Take $q \in Q$ such that $\psi_1(q) = h(a)$. Then $\psi_2 jh(a) = \psi_2\psi_2(q) = 0$, which shows that $\psi_2 jh = 0$, as required.

Now, $h \in \ker(g)$ if and only if $[jh] \equiv 0$ in $\ker(\tilde{\psi_2})/\mathrm{im}(\tilde{\psi_1})$ if and only if $jh = \psi_1 d$, for some $d \in \mathrm{Hom}_R(A,Q)$. If we show that $jh = \psi_1 d$ for $d \in \mathrm{Hom}_R(A,Q)$ if and only if $h = \pi d$, then we have $h \in \ker(g)$ if and only if $h \in \mathrm{im}(\tilde{\pi})$, which is what we want. But this is clear from the definition of j and the fact that $\psi_1(q) = \pi(q)$, for all $q \in Q$.

Finally, to see that g is surjective, take $[l] \in \ker(\tilde{\psi_2})/\mathrm{im}(\tilde{\psi_1})$, where $l \in \mathrm{Hom}_R(A,Q_1)$ satisfies $\psi_2 l = 0$. Thus, $\mathrm{im}(l) \subseteq \mathrm{im}(\psi_1) = C$, so l takes its values in C. Thus, if we let $l' \in \mathrm{Hom}_R(A,C)$ be the map defined by l, then $jl' = l$ and hence $[l] = [jl'] = g(l')$, showing that g is surjective. □

3. Let A, B, C be R-modules. Prove that $\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_R(B,C))$, the so-called Hom-Tensor adjointness. Hint: If $f : A \otimes_R B \to C$, show there exists $g \in \mathrm{Hom}_R(A, \mathrm{Hom}_R(B,C))$ satisfying $g(a)(b) = f(a \otimes b)$, for all $a \in A$ and $b \in B$.

Solution. Fix $f \in \mathrm{Hom}_R(A \otimes B)$. For $a \in A$, define $g(a) : B \to C$ as follows: $g(a)(b) := f(a \otimes b)$, for all $b \in B$. The bilinear properties of tensor product show that $g(a) \in \mathrm{Hom}_R(B,C)$. We next show that the function $g : A \to \mathrm{Hom}_R(B,C)$ given by $a \to g(a)$ is an R-module homomorphism. For $a_1, a_2 \in A$, and $b \in B$ we have

$$g(a_1+a_2)(b) = f((a_1+a_2)\otimes b) = f((a_1 \otimes b) + (a_2 \otimes b)) = f(a_1 \otimes b) + f(a_2 \otimes b) = g(a_1)(b) + g(a_2)(b) = (g(a_1) + g(a_2))(b),$$

showing $g(a_1 + a_2) = g(a_1) + g(a_2)$. Now take $r \in R$, $a \in A$. Then for any $b \in B$,

$$g(ra)(b) = f((ra) \otimes b) = f(r(a \otimes b)) = rf(a \otimes b) = rg(a)(b),$$

showing that $g(ra) = rg(a)$. Therefore, $g \in \mathrm{Hom}_R(A, \mathrm{Hom}_R(B,C))$. We can now define $\phi : \mathrm{Hom}_R(A \otimes B), C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_R(B,C))$ by $\phi(f) = g$.

We now show that ϕ is an isomorphism. Let $f_1, f_1 \in \mathrm{Hom}_R(A \otimes B, C)$, and set $g_1 := \phi(f_1)$ and $g_2 := \phi(f_2)$. Then, for $a \in A$ and $b \in B$, we have, on the one hand

$$\phi(f_1+f_2)(a)(b) = (f_1+f_2)(a \otimes b) = f_1(a \otimes b) + f_2(a \otimes b) = (\phi(f_1)(a) + \phi(f_2)(a))(b).$$

Thus, $\phi(f_1+f_2)(a) = \phi(f_1)(a) + \phi(f_1)(a) = (\phi(f_1) + \phi(f_2))(a)$, for all $a \in A$, and thus, $\phi(f_1+f_2) = \phi(f_1) + \phi(f_2)$. Similarly, one can show that $\phi(rf) = r\phi(f)$, for all $r \in R$ and $f \in \mathrm{Hom}_R(A \otimes B, C)$. Thus, ϕ is an R-module homomorphism. Suppose $\phi(f) = 0$. Then, $\phi(f)(a)(b) = 0$, for all $a \in A$ and $b \in B$. Thus, $f(a \otimes b) = 0$, for all $a \in A$ and $b \in B$, which implies that $f = 0$. Therefore ϕ is one-to-one.

Finally, suppose $g \in \mathrm{Hom}_R(A, \mathrm{Hom}_R(B,C))$. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A \times B & \xrightarrow{h} & A \otimes B \\ & & \downarrow l & \swarrow f & \\ & & C & & \end{array}$$

where $l(a,b) := g(a)(b)$. It is straight forward to check that l is bilinear, so there exists $f : A \otimes B \to C$ satisfying $f(a \otimes b) = g(a)(b)$. This shows that $\phi(f) = g$, so that ϕ is surjective. □

4. Prove that $R/I \otimes_R M \cong M/IM$, for $I \subseteq R$ an ideal and M an R-module. Conclude that for ideals $I, J \subseteq R$, $(R/I) \otimes_R (R/J)$ is isomorphic to $R/(I+J)$ and in particular, $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m \cong \mathbb{Z}_d$, for $d := \mathrm{GCD}(n,m)$.

Solution. Starting the with the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & R/I \times M & \xrightarrow{h} & R/I \otimes M \\ & & \downarrow g & \swarrow f & \\ & & M/IM & & \end{array}$$

where $g(r+I,m) = rm + IM$. It is easy to check that g is bilinear, so there exists $f : R/I \otimes M \to M/IM$ having the property that $f((r+I)\otimes m)) = rm + IM$. We now define $l : M/IM \to R/I \otimes M$ as follows: $l(m+IM) := (1+I)\otimes m$. To see that l is well defined, suppose $m + IM = m' + IM$, so that $m - m' = \sum_j i_i y_j$, with $i_j \in I$ and $y_j \in M$. Then

$$\begin{aligned}(1+I)\otimes m &= (1+I)\otimes(m' + \sum_j i_j y_j)\\ &= (1+I)\otimes m' + (1+I)\otimes \sum_j i_j y_j\\ &= (1+I)\otimes m' + \sum_j ((1+I)i_j)\otimes y_j\\ &= (1+I)\otimes m' + 0 = (1+I)\otimes m',\end{aligned}$$

showing that l is well-defined. Taking $(r+I)\otimes m$ in $R/I\otimes m$, we have

$$lf((r+I)\otimes m) = l(rm+IM) = (1+I)\otimes rm = (1+I)r\otimes m = (r+I)\otimes m.$$

Conversely, for $m + IM \in M/IM$, we have

$$fl(m+IM) = l((1+I)\otimes m) = m + IM,$$

showing that $R/I \otimes M \cong M/IM$.

For the second statement, by the first statement we have

$$(R/I)\otimes(R/J) \cong (R/J)/I(R/J) \cong (R/J)/\{(I+J)/J\} \cong R/(I+J).$$

For the final statement, we apply the second statement to obtain

$$\mathbb{Z}_n \otimes \mathbb{Z}_m = (\mathbb{Z}/n\mathbb{Z})\otimes \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n\mathbb{Z}+m\mathbb{Z}) = \mathbb{Z}/\langle n,m\rangle \cong \mathbb{Z}/d\mathbb{Z},$$

since $\langle n,m\rangle = d\mathbb{Z}$, where $d = \mathrm{GCD}(n,m)$.

5. Prove that if $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of R-modules, and D is a projective R-module, then $0 \to A\otimes D \overset{f\otimes 1_D}{\to} B\otimes D \overset{g\otimes 1_D}{\to} C\otimes D \to 0$ is exact.

Solution. Since D is projective there is a free R-module F and and R-module K such that $F \cong P\bigoplus K$. We first note that $0 \to A\otimes F \to B\otimes F \to C\otimes F \to 0$ is exact, and for this we just need to show that $f\otimes 1_F : A\otimes F \to B\otimes F$ is injective. Without loss of generality, we assume that $P, K \subseteq F$. We may assume $F = \bigoplus_{i\in I} R$, a direct sum of $|I|$ copies of R. Then it is easy to check (you should do this) that $A\otimes F \cong \bigoplus_i A$ and $B\otimes F \cong \bigoplus_i B$. Then, we have the following commutative diagram

$$\begin{array}{ccc} A\otimes F & \xrightarrow{f\otimes 1_F} & B\otimes F\\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta}\\ \bigoplus_i A & \xrightarrow{j} & \bigoplus_i B,\end{array}$$

where α, β are isomorphisms and j is the injective map induced by f. It follows that $f\otimes 1_F$ is injective. We thus obtain the commutative diagram

$$\begin{array}{ccc} A\otimes(P\bigoplus K) & \xrightarrow{f\otimes 1_F} & B\otimes(P\bigoplus K)\\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta}\\ (A\otimes P)\bigoplus(A\otimes K) & \xrightarrow{u} & (B\otimes P)\bigoplus(B\otimes K)\\ \downarrow{\scriptstyle\pi_A} & & \downarrow{\scriptstyle\pi_B}\\ A\otimes P & \xrightarrow{f\otimes 1_P} & B\otimes P\end{array}.$$

In this latter diagram, α, β are isomorphisms, so that u is injective, since $f\otimes 1_F$ is injective. π_A, π_B are the canonical projections. Suppose $(f\otimes 1_F)(x) = 0$, for $x \in A\otimes P$. Then $u(x,0) = (0,0)$, since $u(x,0) = ((f\otimes 1_P)(x), (f\otimes 1_K)(0))$. But u is 1-1, so $x = 0$, which gives what we want. □

6. Let A, B be R-modules and $S \subseteq R$ a multiplicatively closed set. Prove that there is an isomorphism of R_S-modules $\mathrm{Tor}_n^R(A,B)_S \cong \mathrm{Tor}_n^{R_S}(A_S, B_S)$, for all $n \geq 1$. You may use the fact that $(A\otimes_R B)_S \cong A_S \otimes_{R_S} B_S$, as R_S-modules.

Solution. Let $\mathcal{P}: \quad \cdots \to P_2 \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\pi} A \to 0$ be a projective resolution of A. To calculate $\mathrm{Tor}_n^R(A,B)$, we calculate the nth homology of $\mathcal{P}\otimes B$, after dropping the A term. Thus, $\mathrm{Tor}_n^R(A,B) = \ker(\phi_1\otimes 1_B)/\mathrm{im}(\phi_{n+1}\otimes 1_B)$. Therefore,

$$\begin{aligned}\mathrm{Tor}_n^R(A,B)_S &= \{\ker(\phi_n\otimes 1_B)/\mathrm{im}(\phi_{n+1}\otimes 1_B)\}_S\\ &= \ker(\phi_n\otimes 1_B)_S/\mathrm{im}(\phi_{n+1}\otimes 1_B)_S\\ &= \ker((\phi_n)_S\otimes(1_B)_S)/\mathrm{im}((\phi_{n+1})_S\otimes(1_B)_S).\end{aligned}$$

Since $\mathcal{P}_S$ is a projective resolution of B_S, this gives what we want.

7. Calculate $\mathrm{Tor}_n^{\mathbb{Z}_{48}}(\mathbb{Z}_{12}, \mathbb{Z}_{16})$ in two ways, for all $n \geq 0$.

Solution. We set $R := \mathbb{Z}_{48}$ and start with a projective resolution of $\mathbb{Z}_{12} = R/12R$:

$$\cdots \xrightarrow{\cdot 4} R_3 \xrightarrow{\cdot 12} R_2 \xrightarrow{\cdot 4} R_1 \xrightarrow{\cdot 12} R_0 \xrightarrow{\pi} R/12R \to 0,$$

where R_i denotes R in homological degree i. Using the fact that $R \otimes \mathbb{Z}_{16} \cong \mathbb{Z}_{16}$, if we drop $\mathbb{Z}_{12}$ and tensor with $\mathbb{Z}_{16}$ we obtain the complex

$$\cdots \xrightarrow{\cdot 4} \mathbb{Z}_{16} \xrightarrow{\cdot 12} \mathbb{Z}_{16} \xrightarrow{\cdot 4} \mathbb{Z}_{16} \xrightarrow{\cdot 12} \mathbb{Z}_{16} \to 0.$$

The kernel of the map $\mathbb{Z}_{16} \xrightarrow{\cdot 12} \mathbb{Z}_{16}$ is $4\mathbb{Z}_{16}$, since if $12a = 16b$ in $\mathbb{Z}$, $3a = 4b$, so a is divisible by 4. Thus the complex above is exact in odd homological degrees. On the other hand, in even homological degrees of 2 or more, the kernel of the map $\mathbb{Z}_{16} \stackrel{\cdot 4}{\to} \mathbb{Z}_{16}$ is $4\mathbb{Z}_{16}$, while the image is $12 \cdot \mathbb{Z}_{16} = (12\mathbb{Z} + 16\mathbb{Z})/16\mathbb{Z} = 4\mathbb{Z}/16\mathbb{Z} = 4 \cdot \mathbb{Z}_{16}$, these homology modules are also 0. Thus, $\text{Tor}_j^R(\mathbb{Z}_{12}, \mathbb{Z}_{16}) = 0$, for $j \geq 1$. On the other hand the degree zero homology module on the right is just $\mathbb{Z}_{12} \otimes_{\mathbb{Z}_{48}} \mathbb{Z}_{16}$. To calculate this, we note that this is

$$(R/12R) \otimes_R (R/16R) \cong R/\langle 12, 16\rangle \cong R/4R \cong \mathbb{Z}_4.$$

We now take a projective resolution of $\mathbb{Z}_{16}$:

$$\cdots \xrightarrow{\cdot 3} R_3 \xrightarrow{\cdot 16} R_2 \xrightarrow{\cdot 3} R_1 \xrightarrow{\cdot 16} R_0 \xrightarrow{\pi} \mathbb{Z}_{16} \to 0,$$

Tensoring with $\mathbb{Z}_{12}$ and dropping the $\mathbb{Z}_{16}$ term we have the complex

$$\cdots \xrightarrow{\cdot 3} \mathbb{Z}_{12} \xrightarrow{\cdot 16} \mathbb{Z}_{12} \xrightarrow{\cdot 3} \mathbb{Z}_{12} \xrightarrow{\cdot 16} \mathbb{Z}_{12} \to 0.$$

The zeroth homology in the complex above is $\mathbb{Z}_{12}/16\mathbb{Z}_{12} = \mathbb{Z}_{12}/4\mathbb{Z}_{12} \cong \mathbb{Z}_4$, which agrees with our previous calculation. For $n > 0$ and n odd, we have $\text{Tor}_n^R(\mathbb{Z}_{16}, \mathbb{Z}_{12}) = 3\mathbb{Z}_{12}/3\mathbb{Z}_{12} = 0$, since the kernel of the map $\mathbb{Z}_{12} \stackrel{\cdot 16}{\to} \mathbb{Z}_{12}$ is $3\mathbb{Z}_{12}$. For n even, $\text{Tor}_n^R(\mathbb{Z}_{16}, \mathbb{Z}_{12}) = 4\mathbb{Z}_{12}/16\mathbb{Z}_{12} = 4\mathbb{Z}_{12}/4\mathbb{Z}_{12}$, since the kernel of the map $\mathbb{Z}_{12} \stackrel{\cdot 3}{\to} \mathbb{Z}_{12}$ is $4\mathbb{Z}_{12}$.

8. Let $p \in \mathbb{Z}$ be prime and consider the sequence of $\mathbb{Z}$-modules and $\mathbb{Z}$-module homomorphisms $\mathbb{Z}_p \stackrel{\cdot p}{\to} \mathbb{Z}_{p^2} \stackrel{\cdot p}{\to} \mathbb{Z}_{p^3} \stackrel{\cdot p}{\to} \mathbb{Z}_{p^4} \stackrel{\cdot p}{\to} \cdots$. Show that this is a directed system of $\mathbb{Z}$-modules whose direct limit is isomorphic to $\mathbb{Z}_{p^\infty}$.

Solution. For this we set $I := \{1, 2, 3, \ldots, \}$, $M_i := \mathbb{Z}_{p^i}$ and $f_{ij} : M_i \to M_j$ to be multiplication by p^{j-i}, for $i \leq j$. Then we clearly have a direct system of $\mathbb{Z}$-module and $\mathbb{Z}$-module homomorphisms. In each of the modules $\varinjlim M_i$, $\mathbb{Z}_{p^i}$ and $\mathbb{Z}_{p^\infty}$, the elements are all equivalence classes, so we denote $[x]$ for an element of $\varinjlim M_i$, $\overline{a}$ for an element of $M_i := \mathbb{Z}_{p^i}$ and $\frac{a}{p^t}$ for an element of $\mathbb{Z}_{p^\infty}$. For each i, define $g_i M_i \to \mathbb{Z}_{p^\infty}$ by $g_i(\overline{a}) := \frac{a}{p^i}$. This is well defined if we take $a < p^i$. To see that g_i is an R-module homomorphism, suppose $\overline{b} \in M_i$, with $b < p_i$. write $\overline{a+b} = \overline{c}$, with $c < p^i$. Then

$$g_i(\overline{a} + \overline{b}) = g_i(\overline{c}) = \frac{c}{p^i} = \frac{a + b - p^i}{p_i} = \frac{a}{p^i} + \frac{b}{p^i} = g_i(\overline{a}) + g_i(\overline{b}).$$

The argument showing $g_i(r \cdot \overline{a}) = rg_i(\overline{a})$, for $r \in \mathbb{Z}$ is similar, but easier.

Now, for $i \leq j$,

$$g_j f_{ij}(\overline{a}) = g_j(p^{j-i}\overline{a}) = g_j(\overline{p^{j-i}a}) = \frac{p^{j-i}a}{p^j} = \frac{a}{p^i} = g_i(\overline{a}),$$

so $g_i = g_j \circ f_{ij}$, for all $i \leq j$. Thus, by the previous problem, there exists a (unique) $\mathbb{Z}$-module homomorphism $\phi : \varinjlim M_i \to \mathbb{Z}_{p^\infty}$ given by $\phi([\overline{a}]) = \frac{a}{p^i}$, if $\overline{a} \in M_i$. Since $a < p^i$, $\frac{a}{p^i} \neq 0$, unless $\overline{a} = \overline{0}$, so ϕ is 1-1. Moreover, given $fracbp^j \in \mathbb{Z}_{p^\infty}$, we may assume that $b < p^j$, so that if we take $\overline{b} \in M_j$, then $\phi([\overline{b}]) = \frac{b}{p^j}$, showing that ϕ is surjective and hence an isomorphism.

9. Let A be an R-module and suppose $x \in R$ is a non-zerodivisor on both R and A. Suppose

$$\mathcal{F}: \quad \cdots \to F_2 \stackrel{\phi_2}{\to} F_1 \stackrel{\phi_1}{\to} F_0 \stackrel{\pi}{\to} A \to 0$$

is a free resolution of A.

(i) Show directly that the induced sequence

$$\mathcal{F}/x\mathcal{F}: \quad \cdots \to F_2/xF_2 \stackrel{\phi_2'}{\to} F_1/xF_1 \stackrel{\phi_1'}{\to} F_0/xF_0 \stackrel{\pi'}{\to} A/xA \to 0$$

is exact.

(ii) Conclude that $\text{Tor}_n^R(A, R/xR) = 0$, for all $n \geq 1$.

(iii) Now reprove (i) by recalculating $\text{Tor}_n^R(A, R/xR)$, for all $n \geq 1$.

Solution. For (i), clearly $\mathcal{F}/x\mathcal{F}$ is a complex and π' is surjective. Suppose $\pi'(v') = 0$, for $v' \in F_0/xF_0$, the image of $v \in F_0$. Then $\pi(v) = xa$, for some $a \in A$. Since π is surjective, $a = \pi(w)$, for some $w \in F_0$. Thus, $\pi(v) = \pi(xw)$, so that $\pi(v - xw) = 0$. Thus, $v - xw = \phi_1(u)$, for some $u \in F_1$. Thus, $v - \phi_1(u) \in xF_0$, so that $\phi_1'(u') = v'$ in F_0/xF_0, where u' is the image of u in F_1/xF_1. Thus, exactness holds at F_0/xF_0 in $\mathcal{F}/x\mathcal{F}$.

Now suppose $\phi_1'(y') = 0$, for $y' \in F_1/xF_1$. Then $\phi_1(y) = xz$, for some $z \in F_0$. Therefore, $0 = \pi(xz) = x\pi(z)$. Since x is a non-zerodivior on A, $\pi(z) = 0$. Thus, $z \in \ker(\pi) = \text{im}(\phi_1)$, so that $z = \phi_1(t)$, for $t \in F_1$. It follows that $[hi_1(y) = \phi_1(xt)$, so that $\phi_1(y - xt) = 0$. Thus, $y - xt = \phi_2(h)$, for $h \in F_2$. Thus, in F_1/xF_1, we have $y' = \phi_2'(h')$, so exactness in $\mathcal{F}/x\mathcal{F}$ holds at F_1/xF_1. The proof proceeds in similar fashion for exactness at each F_n/xF_n, only for $n \geq 2$, one uses that x is a no-zerodivsor on each F_n by virtue of being a non-zerodivisor on R.

(ii) follows immediately from (i), since $\mathcal{F}/x\mathcal{F}$ is obtained by tensoring $\mathcal{F}$ with R/xR. For (iii), we may calculate $\mathrm{Tor}_n^R(A, R/xR)$ by taking a projective (or free) resolution of R/xR and tensoring with A. Since $0 \to R \stackrel{\cdot x}{\to} R \to R/xR \to 0$ is a free resolution of R/xR, tensoring with A and dropping the A/xA term on the right shows that the Tor modules are the homology modules in the complex $0 \to A \stackrel{\cdot x}{\to} A \to 0$. Since x is a non-zerodivisor on A, it follows that this complex has zero. homology in all homological degrees greater than zero, i.e., $\mathrm{Tor}_n^R(A, R/xR) = 0$, for all $n \geq 1$. $\square$

10. An R-module F is said to be *flat* if whenever $0 \to C \to D \to E \to 0$ is a short eact sequence of R-modules, then the induced sequence $0 \to C \otimes F \to D \otimes F \to E \otimes F$ is exact. Let A be an R-module. A *flat resolution* of A is an exact sequence of the form

$$\mathcal{F}: \quad \cdots \to F_2 \to F_1 \to F_0 \to A \to 0$$

such that each F_j is a flat R-module. The *flat dimension* of A is the least $n \geq 0$ (if it exists) such that A has a flat resolution of length n. For example, a flat module has flat dimension zero. For an R-module A, prove that the following conditions are equivalent:

(i) A has flat dimension less than or equal to n.
(ii) $\mathrm{Tor}_j^R(A, B) = 0$, for all $j \geq n+1$ and all R-modules B.
(iii) $\mathrm{Tor}_{n+1}^R(A, B) = 0$, for all R-modules B.

Solution. We begin with three observations. The first is that if $\mathcal{M}: \cdots \to M_2 \stackrel{f_2}{\to} M_1 \stackrel{f_1}{\to} M_0 \to 0$ is an exact sequence of R-modules and A is a flat R-module, then tensoring $\mathcal{M}$ with A preserves exactness. To see this, let K be the kernel of f_1, so that $0 \to K \stackrel{i}{\to} M_1 \stackrel{f_1}{\to} M_0 \to 0$ is exact. Thus, $0 \to K \otimes A \stackrel{i \otimes 1_A}{\to} F_1 \otimes A \stackrel{f_1 \otimes 1_A}{\to} M_0 \otimes A \to 0$ is exact. This shows that $K \otimes A$ is the kernel of $f_1 \otimes 1_A$. On the other hand, since f_2 maps M_2 onto K, $f_2 \otimes 1_A$ maps $M_2 \otimes A$ onto $K \otimes A = \ker(f_1 \otimes 1_A$. In other words, the sequence $M_2 \stackrel{f_1 \otimes 1_A}{\to} M_1 \otimes A \stackrel{f_1 \otimes 1_A}{\to} M_0 \otimes A$ is exact. This argument can be repeated inductively to show $\mathcal{M} \otimes A$ is exact. The second observation is that if in the flat resolution $\mathcal{F}$ above, K is the kernel of the map from F_0 to A, then for $n \geq 1$, the flat dimension of A is less than or equal to n if and only if the flat dimension of K is less than or equal to $n-1$. This follows immediately from the definition of flat dimension. For the third observation, let K be as in the second observation, so that we have the exact sequence $0 \to K \to F_0 \to A \to 0$. Tensoring with B yields the long exact sequence in Tor

$$\cdots \mathrm{Tor}_{j+1}^R(F_0, B) \to \mathrm{Tor}_{j+1}^R(A, B) \to \mathrm{Tor}_j^R(K, B) \to \mathrm{Tor}_j^R(F_0, B) \to \cdots$$

By the first observation, $\mathrm{Tor}_j^R(F_0, B) = 0$, for all $j \geq 1$, so that $\mathrm{Tor}_j^R(K, B) \cong \mathrm{Tor}_{j+1}^R(A, B)$, for all $j \geq 1$, showing that dimension shifting holds for a flat presentation.

To prove the equivalence of (i)-(iii), we induct on n. Suppose $n = 0$, so that A is a flat R-module. The (ii) follows from (i) by the first observation above. Clearly (ii) implies (iii). Now suppose (iii) holds. Take $0 \to D \to C \to B \to 0$ any short exact sequence of R-modules. By the long exact Tor sequence, we have

$$\cdots \to \mathrm{Tor}_1^R(A, B) \to D \otimes A \to C \otimes A \to B \otimes A \to 0,$$

which by (iii) shows that the sequence $0 \to D \otimes A \to C \otimes A \to B \otimes A \to 0$ is exact, and thus (iii) implies (i).

No suppose $n > 0$ and (i)-(iii) hold for $n-1$. Suppose A has flat dimension less than or equal to n so that by the second observation, K has flat dimension less than or equal to $n-1$. Thus, by induction and the third observation, $0 = \mathrm{Tor}_j^R(K, B) = \mathrm{Tor}_{j+1}^R(A, B)$, for all $j \geq n$, which gives (ii). Since (ii) implies (iii) is trivial, assume (iii). Then, by dimension shifting, $\mathrm{Tor}_n^R(K, B) = 0$, so by induction K has flat dimension less than or equal to $n-1$, and thus A has flat dimension less than or equal to n $\square$.